ISRAEL JOURNAL OF MATHEMATICS 81 (1993), 193–202

WEIGHTS AND L_Φ-BOUNDEDNESS OF THE POISSON INTEGRAL OPERATOR

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ABSTRACT

In this paper, we get a necessary and sufficient condition on the weights (μ, v) for the Poisson integral operator to be bounded from $L_\Phi(R^n, v(x)dx)$ to weak-$L_\Phi(R^{n+1}_+, d\mu)$, where Φ is an N-function satisfying the Δ_2-condition. We also find a necessary and sufficient condition on the weights (μ, v) for the Poisson integral operator to be bounded from $L_\Phi(R^n, v(x)dx)$ to $L_\Phi(R^{n+1}_+, d\mu)$ under some additional condition.

1. Introduction

Let P denote the following Poisson integral operator:

$$P(f)(x,t) = \int_{\mathbb{R}^n} f(y)p(x-y,t)dy \quad (x \in \mathbb{R}^n,\ t > 0)$$

where

$$p(x,t) := \frac{C_n t}{(|x|^2 + t^2)^{(n+1)/2}}.$$

Let Φ be an N-function on $[0,\infty)$, i.e., $\Phi(t) = \int_0^t \varphi(t)dt$ where $\varphi:\ [0,\infty) \to \mathbb{R}^1$ is continuous from the right, non-decreasing on $[0,\infty)$, $\varphi(s) > 0$ for $s > 0$, $\varphi(0) = 0$ and $\varphi(+\infty) = +\infty$. In this paper, we shall consider the following two questions:

Q-1: *For a given nonnegative measure μ on $\mathbb{R}^{n+1}_+$ and a weight v on $\mathbb{R}^n$, what are the conditions on (μ, v) for P to be bounded from $L_\Phi(\mathbb{R}^n, v(x)dx)$ to weak-$L_\Phi(\mathbb{R}^{n+1}_+, d\mu)$?*

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* Partially supported by NNSF of P.R. China

Received May 20, 1991

Q-2: *What are the conditions on* (μ, v) *for* P *to be bounded from* $L_\Phi(\mathbb{R}^n, v(x)dx)$ *to* $L_\Phi(\mathbb{R}^{n+1}_+, d\mu)$*?*

This kind of problem was originally studied by Carleson [1] (for $v = 1$, $\Phi(t) = t^p$, $1 < p < +\infty$), Fefferman and Stein [2] (for $\Phi(t) = t^p$, $1 < p < +\infty$) and Muckenhoupt [6] (for $\Phi(t) = t^p$, $1 < p < +\infty$, $d\mu(x,t) = u(x)dx \otimes d\delta_0(t)$ where $\delta_0(t)$ denotes the Dirac measure at 0). The problems were proposed and studied in such a unified form in Ruiz [7] and Ruiz–Torrea [8].

For the above questions, it is enough to consider the following maximal function $\mathcal{M}$ instead of P since $\mathcal{M}(f)$ and $P(f)$ are comparable with each other for nonnegative f. $\mathcal{M}$ is defined by

$$\mathcal{M}(f)(x,t) = \sup_{\text{cube}Q\ni x \text{ and } l(Q)\geq t} |Q|^{-1} \int_Q |f(y)|dy$$

where $l(Q)$ denotes the side length of Q. In this paper, "cube" always means the cubes with sides parallel to the coordinate axes.

For Q-1, our result is as follows.

THEOREM 1: *For an* N*-function* Φ *satisfying the* Δ_2*-condition, a nonnegative measure* μ *on* $\mathbb{R}^{n+1}_+$ *and a weight* ν *on* $\mathbb{R}^n$*, the following inequality holds:*

$$(1) \qquad \mu(\{(x,t):\ \mathcal{M}(f)(x,t) > \eta\}) \leq \frac{C_1}{\Phi(\eta)} \int_{\mathbb{R}^n} \Phi(|f|)\nu(x)dx \qquad (\forall \eta > 0)$$

if and only if $(\mu,\nu) \in A^+_\Phi$*, i.e.*

$$(2) \qquad \sup_{\text{cube}Q \text{ and } t>0} \varphi\left(\left(\psi\left(\frac{1}{t\nu}\right)\right)_Q\right) \cdot t\mu(\widetilde{Q})/|Q| = C_2 < \infty$$

where $\widetilde{Q} := Q \times (0, l(Q)]$, $\mu(\widetilde{Q}) := \int_{\widetilde{Q}} d\mu$, $(g)_Q := |Q|^{-1}\int_Q g(x)dx$ *and*

$$(3) \qquad \psi(t) := \sup\{s :\ \varphi(s) \leq t\}.$$

The Δ_2*-condition means that*

$$(3') \qquad \Phi(2t) \leq C_\Phi \Phi(t) \qquad (t > 0).$$

Furthermore, $C^{-1}_{n,\Phi} \leq C_1/C_2 \leq C_{n,\Phi}$ *for the minimal choice of* C_1.

Remark A: **The case when $\Phi(t) = t^p$, $1 < p < \infty$.** For $v = 1$, the equivalence of (1) and (2) was shown by Carleson [1] and (2) is just the so-called Carleson condition, i.e., $\mu(\widetilde{Q}) \le C_\mu |Q|$. And Fefferman and Stein's condition [2]

$$\sup_{x \in Q} \mu(\widetilde{Q}) \le C_{\mu,\nu} v(x) \qquad \text{a.e. } x \in \mathbb{R}^n$$

is stronger than (2) because the last inequality means

$$\inf_{x \in Q} v(x) \le t^{-1} (\varphi((\psi(\frac{1}{tv}))_Q))^{-1}.$$

The general condition on (μ, v) for (1) was found by Ruiz [7], i.e.

$$(2') \qquad \sup_{\text{cube } Q} \big(|Q|^{-1} \int_Q v(x)^{-p'/p} dx\big)^{p/p'} \mu(\widetilde{Q})/|Q| < \infty.$$

∎

Remark B: **The case when $d\mu(x,t) = u(x)dx \otimes d\delta_0(t)$.** For $\Phi(t) = t^p$, $1 < p < +\infty$, the equivalence was proved by Muckenhoupt [6] and (2) is just the A_p-condition, i.e.

$$(4) \qquad \sup_{\text{cube } Q} \big(\frac{1}{|Q|} \int_Q u(x) dx\big) \big(\frac{1}{|Q|} \int_Q v(x)^{-p'/p} dx\big)^{p/p'} < \infty.$$

For general Φ, if Φ and its complementary function

$$(5) \qquad \Psi(t) := \int_0^t \psi(s) ds$$

satisfy (3′) where ψ is defined by (3), the equivalence of (1) and (2) was shown by Gallardo, and (2) is equivalent to the A_Φ-condition (see [3,5]). ∎

For Q-2, our result is partial. We first introduce some notation. Let

$$\mathcal{N}(f)(x,t) = \sup_{\substack{\text{dyadic cube } Q \ni x \\ \text{and } \ell(Q) \ge t}} (|f|)_Q,$$

$$N_\sigma(f)(x) = \sup_{\text{dyadic cube} Q \ni x} (|f|)_{\sigma,Q},$$

$$(g)_{\sigma,Q} = \int_Q g(y)\sigma(y) dy / \sigma(Q),$$

$$\widetilde{N}_\sigma(f)(x) = \sigma(x) N_\sigma(f/\sigma)(x),$$

$$\tau_y(f)(x) = f(x-y).$$

Then we have

THEOREM 2: *Suppose Φ is an N-function satisfying* (3). *If there is a weight σ on $\mathbb{R}^n$ such that*

$$(6)\qquad \sup_{y\in\mathbb{R}^n}\int_{\mathbb{R}^n}\Phi(\tau_y^{-1}\tilde{N}_{\tau_y\sigma}(\tau_y(f)(x))\nu(x)dx \le C_3\int_{\mathbb{R}^n}\Phi(|f|)\nu(x)dx,$$

then $\mathcal{M}$ is $L_\Phi(\mathbb{R}^n,\nu(x)dx)\to L_\Phi(\mathbb{R}_+^{n+1},d\mu)$ bounded, i.e.

$$(7)\qquad \int_{\mathbb{R}_+^{n+1}}\Phi(\mathcal{M}(f)(x,t))d\mu(x,t)\le C_4\int_{\mathbb{R}^n}\Phi(|f|)\nu(x)dx$$

if and only if

$$(8)\qquad \int_{\tilde{Q}}\Phi(\mathcal{M}(\eta\sigma\chi_Q)(x,t))d\mu(x,t)\le C_5\int_Q\Phi(\eta\sigma)\nu(x)dx\qquad(\forall\eta>0).$$

Further, $C_{n,\Phi}^{-1}\le C_4/C_5\le C_{n,\Phi}$.

∎

Remarks: (C) It is easy to see that $\tau_y^{-1}(\tilde{N}_{\tau_y\sigma}(\tau_y(f)))(x)\le\sigma(x)M_\sigma(f/\sigma)(x)$ for any $y\in\mathbb{R}^n$, where M_σ is the Hardy–Littlewood maximal function operator with respect to the measure $\sigma(x)dx$.

(D) If $y\in A_\Phi$, then (6) is true for $\sigma=1$ by [5]. Thus we have

COROLLARY 3: *If $\nu\in A_\Phi$, then $\mathcal{M}$ is bounded from $L_\Phi(\mathbb{R}^n,\nu(x)dx)$ to $L_\Phi(\mathbb{R}_+^{n+1},d\mu)$ iff $\mu(\tilde{Q})\le C_5\nu(Q)$, where Φ and Ψ satisfy* (3).

(E) If $C_\Phi^{-1}\le\Phi(st)/(\Phi(s)\Phi(t))\le C_\Phi$ then (6) is true for any weight v and $\sigma:=\psi(\frac{1}{v})$ where ψ is defined by (3). Actually, N_σ is bounded from $L^\infty(\mathbb{R}^n,\sigma(x)dx)$ to itself and from $L^1(\mathbb{R}^n,\sigma(x)dx)$ to weak-$L^1(\mathbb{R}^n,\sigma(x)dx)$ for any weight σ. So, by Theorem 2.17 of [3], N_σ is $L_\Phi(\mathbb{R}^n,\sigma(x)dx)$-bounded. Thus, by Lemma 1 of the next section, for $\sigma:=\psi(\frac{1}{v})$, we have

$$\int\Phi(\tilde{N}_\sigma(f))v\le C_\Phi\int\sigma(\varphi(\psi(\frac{1}{v}))v)\Phi(N_\sigma(\frac{|f|}{\sigma}))\le C_\Phi\int\sigma\Phi(N_\sigma(\frac{|f|}{\sigma}))$$
$$\le C_{\Phi,n}\int\sigma\Phi(\frac{|f|}{\sigma})\le C_{\Phi,n}\int\sigma\Phi(|f|)/\Phi(\sigma)\le C_{\Phi,n}\int\sigma\Phi(|f|)v.$$

Similar estimates hold for the operator $\tau_y^{-1}\tilde{N}_{\tau_y\sigma}\tau_y$. So, we have

COROLLARY 4: *If $C_\Phi^{-1}\le\Phi(st)/(\Phi(t)\Phi(s))\le C_\Phi$, Φ and its complementary function Ψ satisfy* (3′), *then* (7) *holds iff* (8) *holds.*

(F) In particular, Corollary 4 holds for $\Phi(t)=t^p$, $1<p<+\infty$. In this case, we get Sawyer's result [9] when $d\mu(x,t)=u(x)dx\otimes d\delta_0(t)$ and Ruiz–Torrea's result [8] for general μ. ∎

2. Proof of Theorem 2

Theorem 2 will follow from

THEOREM 2′: *If there is a weight σ on $\mathbb{R}^n$ such that*

$$\int_{\mathbb{R}^n} \Phi(\tilde{N}_\sigma(f)(x))\nu(x)dx \le C_6 \int_{\mathbb{R}^n} \Phi(|f|)\nu(x)dx, \tag{6'}$$

then

$$\int_{\mathbb{R}^{n+1}_+} \Phi(\mathcal{N}(f)(x,t))d\mu(x,t) \le C_7 \int_{\mathbb{R}^n} \Phi(|f|)\nu(x)dx \tag{7'}$$

if and only if

$$\int_{\tilde{Q}} \Phi(\mathcal{N}(\eta\sigma\chi_Q)(x,t))d\mu(x,t) \le C_8 \int_Q \Phi(\eta\sigma)\nu(x)dx \qquad (\forall \eta > 0) \tag{8'}$$

where Φ is an N-function satisfying (3′), and $1 \le C_7/C_8 \le C_\Phi$.

Actually, a "translation" discussion shows that, under the conditions of Theorem 2 (i.e. (6) and (7)), the following inequality holds:

$$\sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^{n+1}_+} \Phi((\tau_y^{-1}\mathcal{N}\tau_y)(f))(x,t)d\mu(x,t) \le C_7 \int_{\mathbb{R}^n} \Phi(|f|)v(x)dx. \tag{9}$$

Thus, by Lemma 2 of [8] and Jensen's inequality, we can easily get (8) from (9).

Now, we shall prove Theorem 2′. The idea is essentially from [4,9]. At first, we have

LEMMA 1: *For any N-function Φ, $t \le \varphi(\psi(t))$ and $\Phi(t) \le t\varphi(t)$. If Φ satisfies (3′), then $\varphi(\psi(t)) \le C_\Phi t$ and $\Phi(t) \ge t\varphi(t)/C_\Phi$.*

Proof: The first part easily follows from the right-hand continuity and monotonicity of φ. Now, if Φ satisfies (3′), we have

$$\Phi(t) \ge C_\Phi^{-1}\Phi(2t) \ge C_\Phi^{-1} \int_t^{2t} \varphi(s)ds \ge C_\Phi^{-1} t\varphi(t),$$

and thus

$$\varphi(\psi(t)) \le \lim_{\alpha\to 0+} \varphi(2(\psi(t)-\alpha)) \le C_\Phi \lim_{\alpha\to 0+} \varphi(\psi(t)-\alpha) \le C_\Phi t. \qquad \blacksquare$$

Now, "(7)⇒(8)" is obvious if we take $f = \eta\sigma\chi_Q$. To prove "(8′) ⇒ (7′)", we need

LEMMA 2: (8′) *implies the following inequality:*

$$(8'') \qquad \int_{\hat{G}_d} \Phi(\mathcal{N}(\eta\sigma\chi_G)(x,t))d\mu(x,t) \le C_8 \int_G \Phi(\eta\sigma)\nu(x)dx \qquad (\forall \eta > 0)$$

where $\hat{G}_d := \bigcup_{\text{dyadic } Q\subset G} \tilde{Q}$.

Proof: Consider $\mathcal{N}^{(R)}(f)$ defined as $\mathcal{N}(f)$ but with the additional restriction $l(Q) \le R$, $R > 0$. Let

$$\mathcal{A}_k^R = \{\tilde{Q} : (\eta\sigma)_Q > 2^k, \ l(Q) \ge R, \ \ Q \subset G\},$$

and choose a maximal subfamily $\{\tilde{Q}_{k,j}\}_j$ from $\mathcal{A}_k^R$ (it is possible because $\sup\{l(Q) : \ \tilde{Q} \in \mathcal{A}_k^R\} \le R < +\infty$). Then it is easy to see that

$$\bigcup_j \tilde{Q}_{k,j} = \bigcup_{\tilde{Q}\in\mathcal{A}_k^R} \tilde{Q} = \{(x,t) : \ \mathcal{N}^{(R)}(\sigma\eta)(x,t) > 2^k, \ x \in G\}$$

and

$$\hat{G}_d \cap (G \times [0,R]) = \bigcup_{k,j} \tilde{Q}_{k,j}.$$

Now, put

$$E_{k,j} = \tilde{Q}_{k,j} - \bigcup_{k,j} \tilde{Q}_{k+1,j},$$

$\{\tilde{Q}_i\}_i$ to be a maximal subfamily of $\{\tilde{Q}_{k,j}\}_{k,j}$. Then

$$\begin{aligned}
\int_{\hat{G}_d} \Phi(\mathcal{N}^{(R)}(\eta\sigma\chi_G)(x,t))d\mu(x,t) &= \int_{\hat{G}_d\cap(G\times[0,R])} \Phi(\mathcal{N}^{(R)}(\eta\sigma\chi_G)(x,t))d\mu(x,t) \\
&\le \sum_{k,j} \Phi(2^{k+1})\mu(E_{k,j}) \le C_\Phi \sum_i \sum_{Q_{k,j}\subset Q_i} \Phi((\eta\sigma)_{Q_{k,j}})\mu(E_{k,j}) \\
&\le C_\Phi \sum_i \sum_{Q_{k,j}\subset Q_i} \int_{E_{k,j}} \Phi(\mathcal{N}^{(R)}(\eta\sigma\chi_{Q_i})(x,t))d\mu(x,t) \\
&\le C_\Phi \int_{\bigcup_i Q_i} \Phi(\eta\sigma)(x)v(x)dx = C_\Phi \int_G \Phi(\eta\sigma)(x)v(x)dx.
\end{aligned}$$

Finally, letting $R \to +\infty$, we get (8″).

Now, we shall prove "(8′) ⇒ (7′)". Similarly to the proof of Lemma 2, let

$$\mathcal{B}_k^R = \{\tilde{Q} : (|f|)_Q > 2^k, \ l(Q) \le R\},$$

$$\{\tilde{Q}_{k,j}\}_j \text{ be a maximal subfamily of } \mathcal{B}_k^R,$$
$$F_{k,j} = \tilde{Q}_{k,j} - \bigcup_j \tilde{Q}_{k+1,j},$$
$$\Gamma(\eta) = \{(k,j):\ (|f|/\sigma)_{\sigma,Q_{k,j}} > \eta\},$$
$$G(\eta) = \bigcup_{(k,j)\in\Gamma(\eta)} \tilde{Q}_{k,j};$$

then,

$$\mathbb{R}_+^{n+1} = \bigcup_{(k,j)} \tilde{Q}_{k,j} = \{(x \in \mathbb{R}^n:\ N_\sigma^{(R)}(f/\sigma)(x) > \eta\} \text{ and } \widehat{G}(\eta)_d \supset \bigcup_{(k,j)\in\Gamma(\eta)} \tilde{Q}_{k,j}$$

where $N_\sigma^{(R)}$ is defined similarly as N_σ but with restriction $l(Q) \le R$. Therefore, we have

$$\begin{aligned}
\int_{\mathbb{R}_+^{n+1}} \Phi(\mathcal{N}^{(R)}(f))d\mu &\le \sum_{(k,j)} \Phi(2^{k+1})\mu(F_{k,j}) \le C_\Phi \sum_{(k,j)} \Phi((|f|)_{Q_{k,j}})\mu(F_{k,j}) \\
&\le C_\Phi \sum_{(k,j)} \Phi((\sigma)_{Q_{k,j}}(|f|/\sigma)_{\sigma,Q_{k,j}})\mu(F_{k,j}) \\
&\le C_\Phi \sum_i \sum_{(k,j)\in\Gamma(2^i)-\Gamma(2^{i+1})} \Phi((\sigma)_{Q_{k,j}}2^{k+1})\mu(F_{k,j}) \\
&\le C_\Phi \sum_i \sum_{(k,j)\in\Gamma(2^i)} \int_{F_{k,j}} \Phi(2^i\mathcal{N}^{(R)}(\sigma\chi_{Q_{k,j}}))d\mu(x,t) \\
&\le C_\Phi \sum_i \int_{\widehat{G(\eta)}_d} \Phi(2^i\mathcal{N}^{(R)}(\sigma\chi_{G(2^i)}))d\mu(x,t) \\
&\le C_\Phi \sum_i \int_{G(2^i)} \Phi(2^i\sigma)v(x)dx \quad (\text{by } (8'')) \\
&= C_\Phi \int_{\mathbb{R}^n} \Big(\sum_{i:\ 2^i\le N_\sigma^{(R)}(f/\sigma)} \Big)\Phi(2^i\sigma)v(x)dx \quad (\text{by } (8'')) \\
&\le C_\Phi \int_{\mathbb{R}^n} \Phi(\sigma N_\sigma^{(R)}(f/\sigma))v(x)dx
\end{aligned}$$

because, by Lemma 1,

$$\begin{aligned}
\sum_{i\le\alpha} \Phi(2^i\sigma) &\le C_\Phi \sum_{i\le\alpha} 2^i\sigma\varphi(2^i\sigma) \\
&\le C_\Phi 2^i\sigma\varphi(2^\alpha\sigma)\sum_{i\le\alpha} 2^i \le C_\Phi 2^\alpha\sigma\varphi(2^\alpha\sigma) \le C_\Phi\Phi(2^\alpha\sigma).
\end{aligned}$$

Combining the above with (6′), we get (7′). ∎

3. Proof of Theorem 1

Proof of "(1) ⇒ (2)": Taking $f = \chi_Q \psi(\frac{1}{v})$ and $\eta_Q = (f)_Q$, we get

$$\mathcal{M}(f)(x,t) \geq (f)_Q \qquad \text{for any } (x,t) \in \tilde{Q}$$

and

$$\mu(\{(x,t):\ \mathcal{M}(f)(x,t) \geq \eta_Q\}) \leq (C_1/\Phi(\eta_Q)) \int_Q \Phi(|f|)v(x)dx,$$

i.e.

$$\begin{aligned}\mu(\tilde{Q})/|Q| &\leq C_1(\Phi(\psi(\frac{1}{v})v))_Q/\Phi(\eta_Q)\\ &\leq C_1C_\Phi(\psi(\frac{1}{v})v\varphi(\psi(\frac{1}{v})))_Q/((\psi(\frac{1}{v}))_Q\varphi((\psi(\frac{1}{v}))_Q)) \leq C_1C_\Phi/\varphi((\psi(\frac{1}{v}))_Q)\end{aligned}$$

by Lemma 1. ∎

Proof of "(2) ⇒ (1)": At first, we claim that (2) implies

$$\mu(\tilde{Q})/|Q| \leq C_2C_\Phi(\Phi(|f|)v)_Q/\Phi((|f|)_Q). \tag{10}$$

Taking (10) for granted, we can prove (1) easily. Let $\mathcal{M}^{(R)}$ be the maximal operator defined as $\mathcal{M}$, but with restriction $l(Q) \leq R$ in the defining identity of $\mathcal{M}$. Put

$$\begin{aligned}\Omega_\eta &= \{(x,t) \in \mathbb{R}^{n+1}_+ :\ \mathcal{M}^{(R)}(f)(x,t) > \eta\},\\ \Omega'_\eta &= \{x \in \mathbb{R}^n :\ \mathcal{M}^{(R)}(f)(x,0) > \eta\},\\ t_{R,\eta}(x) &= \sup\{t:\ (x,t) \in \Omega_\eta\} \qquad (\leq R \ \text{ for any } x \in \Omega'_\eta),\end{aligned}$$

then $\{Q_{x,R,\eta}\}_{x\in\Omega'_\eta}$ is a covering of Ω'_η, where $Q_{x,R,\eta}$ is a cube containing x, having side length $t_{R,\eta}(x)$ and satisfying $(f)_{Q_{x,R,\eta}} > \eta$. Because of the finiteness of $\sup\{l(Q_{x,R,\eta}) : x \in \Omega'_\eta\}$, by a Besicovitch-type covering lemma, we can choose a subfamily of $\{Q_j\}_j$ such that

$$\bigcup_j Q_j \supset \Omega'_\eta \supset \Omega'_{3^n\eta} \quad \text{and} \quad \sum_j \chi_{Q_j} \leq C_n \chi \bigcup_j Q_j.$$

Then, obviously, $(f)_{Q_j} \geq \eta$ and $\{\tilde{Q}_j\}_j$ is a covering of $\Omega_{3^n\eta}$ because for any $(x,t) \in \Omega_{3^n\eta}$, $x \in$ some Q_j, say, $Q_{y,R,\eta}$, and $t_{y,R,\eta} > t_{y,R,3^n\eta} \geq t$ for, otherwise, $3Q_{x,R,3^n\eta} \supset Q_{y,R,\eta}$ and thus

$$\begin{aligned}\mu(\Omega_{3^n\eta}) &\leq \mu(\bigcup_j \tilde{Q}_j) \leq \sum_j \mu(\tilde{Q}_j) \leq C_2C_\Phi \sum_j |Q_j|(\Phi(|f|)v)_{Q_j}/\Phi((|f|)_{Q_j})\\ &\leq (C_2C_\Phi/\Phi(\eta)) \sum_j \int_{Q_j} \Phi(|f|)v(x)dx \leq (C_2C_{n,\Phi}/\Phi(\eta)) \int_{\mathbb{R}^n} \Phi(|f|)v(x)dx.\end{aligned}$$

Finally, we prove our "claim", i.e. (2) implies (10). Let

$$\| f\|_{\Phi,w} = \inf\{\eta : \int_{\mathbb{R}^n} \Phi(|f|/\eta)w(x)dx \leq \Phi(1)\}.$$

It is well-known that

$$\Big|\int_{\mathbb{R}^n} g(x)f(x)dx\Big| \leq \| f\|_{\Phi,w}\| g\|_{\Psi,w}$$

where Ψ is the complementary function of Φ. Now, by Lemma 1

$$\int_Q \Psi(\frac{1}{v\eta\alpha})\alpha v dx \leq \int_Q \Psi(\frac{1}{v\eta\alpha})\frac{1}{\eta}dx \leq \psi(C_2|Q|/(\eta\alpha\mu(\tilde{Q})))|Q|/\eta \quad \text{(by (2))}$$

so

$$\int_Q \Psi(\frac{1}{v\eta\alpha})\alpha v dx \leq \Psi(1)$$

if

$$\eta \geq C_\Phi\Phi^{-1}(1/(\alpha\mu(\tilde{Q}))\psi(C_2|Q|/(\eta\alpha\mu(\tilde{Q})))|Q|/\eta \leq 1/C_\Phi. \tag{11}$$

On the other hand, by Lemma 1, (2) implies that if

$$\eta \geq C_\Phi|Q|\Phi^{-1}(1/\alpha\mu(\tilde{Q}))$$

where Φ^{-1} is the inverse function of Φ, then

$$1/\mu(\tilde{Q}) \leq \Phi(C_\Phi\eta/(C_\Phi|Q|))$$

which means

$$C_2|Q|/(\eta\alpha\mu(\tilde{Q})) \leq \varphi(|Q|/(\eta C_\Phi)C_\Phi) \quad \text{(by Lemma 1)}.$$

Again, by Lemma 1, the last inequality implies (11). Thus

$$\| \chi_Q(\alpha v)^{-1}\|_{\Psi,\alpha v} \leq C_\Phi|Q|\Phi^{-1}(1/\alpha\mu(\tilde{Q}))$$

and

$$\begin{aligned}(|f|) &\leq \| f\chi_Q\|_{\Psi,\alpha v}\| \chi_Q(\alpha v)^{-1}\|_{\Psi,\alpha v} \\ &\leq C_\Phi|Q|\Phi^{-1}(1/\alpha\mu(\tilde{Q})) = C_\Phi|Q|\Phi^{-1}(1/(\mu(\tilde{Q})))\end{aligned}$$

for $\alpha = \big(\int_Q \Phi(|f|)v(x)dx\big)^{-1}$. Therefore, having (3′), we get

$$\begin{aligned}(\Phi(|f|)v)_Q/\Phi((|f|)_Q) &\geq (\Phi(|f|)v)_Q/\Phi(C_\Phi\Phi^{-1}((\Phi(|f|)v)_Q|Q|/\mu(\tilde{Q}) \\ &\geq C_\Phi(\Phi(|f|)v)_Q/((\Phi(|f|)v)_Q|Q|/\mu(\tilde{Q}) = C_\Phi\mu(\tilde{Q})/|Q|. \quad \blacksquare\end{aligned}$$

Final Remark: (G) It would be interesting to find out a necessary and sufficient condition on (μ, v) for the validity of (7) without the restrictive condition (6). ∎

ACKNOWLEDGEMENT: This work was done when I visited Kiel University during February – July, 1991. I would like to express my many thanks to Prof. Dr. A. Irle for his hospitality and to Prof. Dr. H. König for showing me papers of J. Bourgain, J.L. Rubio de Francia, F. Ruiz, J.L. Torrea and others which stimulated this work.

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